Δps = pressure drop owing to skin friction, lb.$_f$/sq. ft.
q = rate of heat transfer, B.t.u./hr.
T_b = temperature of water entering heat transfer section, °F.
T_w = average temperature of wall of heat transfer section, °F.
$\bar{U}$ = average velocity of flow, ft./sec.
$\bar{U}_D$ = average velocity of flow based on tube diameter D
$\bar{U}_{D-2e}$ = average velocity of flow based on tube diameter, $D - 2e$, ft.
X = length of measuring section, ft.
x = direction parallel to pipe axis (also used for length of smooth pipe — ft.)
y = direction perpendicular to pipe axis

Greek Letters

$\bar{\tau}$ = average resisting stress at the wall, lb.$_f$/sq. ft.
ρ = water density, lb.$_m$/cu. ft.
ν = kinematic viscosity, sq. ft./sec.
μ_w = viscosity of water at temperature T_w, lb.$_m$/ft. sec.
μ_b = viscosity of water at temperature T_b, lb.$_m$/ft. sec.

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COMMUNICATIONS TO THE EDITOR

Multicomponent Diffusion in Restricted Systems

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In an earlier communication (*1*), the authors presented a generalization of the Arnold semi-infinite problem (*2*) to multicomponent systems, starting with the flux equations of Onsager (*3*). The techniques used in that presentation are extended here to the case of unsteady state restricted diffusion. In this case, the system consists of two identical cylindrical compartments, originally filled with uniform multicomponent solutions of slightly different compositions and connected end to end at zero time (the Loschmidt experiment). The results obtained are of significant practical utility; for example, they allow extension of the method of Harned and Nuttal (*4*) to the determination of multicomponent diffusivities.

Of considerably more importance is the method for solution of this example. It allows extension of many descriptions of binary diffusion to the corresponding multicomponent operations. Essentially, the binary method is paralleled; the principal additional step is the separation of a set of simultaneous differential equations.

To begin, the multicomponent flux equation with a reference frame of volume average velocity is inserted into the equation of continuity:

$$\frac{\partial \rho_i}{\partial t} = \nabla \cdot \left[\sum_{j=1}^{r-1} D^o_{ij} \nabla \rho_j \right] - \nabla \cdot \rho_i \mathbf{v}^o \tag{1}$$

If constant diffusion coefficients, constant partial molar volumes, and a one-dimensional system are assumed, Equation (1) may be simplified to

$$\frac{\partial \rho_i}{\partial t} = \sum_{j=1}^{r-1} D^o_{ij} \frac{\partial^2 \rho_j}{\partial z^2} \tag{2}$$

The boundary conditions for the system described above are

$$t = 0 \quad \begin{array}{ll} 0 < z < b & \rho_i = \rho_{i+} \\ -b < z < 0 & \rho_i = \rho_{i-} \end{array} \tag{3}$$

$$t > 0 \qquad z = \pm b \qquad \frac{\partial \rho_i}{\partial z} = 0 \tag{4}$$

To solve this equation, a solution is postulated

$$\frac{\rho_i - \bar{\rho}_i}{\rho_{i+} - \bar{\rho}_i} = f(z)\, g_i(t) \tag{5}$$

where

$$\bar{\rho}_i = \frac{1}{2} (\rho_{i+} + \rho_{i-}) \tag{6}$$

The binary solution with a z dependence of the same functional form for both components suggests this assumption which allows separation of Equation (2); $f(z)$ and $g_i(t)$ are easily found (*5*).

The solution has the form

$$\frac{\rho_i - \bar{\rho}_i}{\rho_{i+} - \bar{\rho}_i} = 2 \sum_{n=0}^{\infty} \left[\frac{\sin\left(\alpha_n \frac{z}{b}\right)}{\alpha_n} \sum_{k=1}^{r-1} B_{ik}\, e^{-\sigma_k \left(\frac{\alpha_n}{b}\right)^2 t} \right] \tag{7}$$

where

$$\alpha_n = \left(n + \frac{1}{2}\right)\pi \tag{8}$$

and the σ_k are the eigenvalues of the characteristic equation; that is, they are the roots of the determinant of Δ, where

$$\Delta_{ij} = D^o_{ij} - \sigma\, \delta_{ij} \tag{9}$$

All roots of this characteristic equation are real and are a complete set even when they are equal (*3*). The B_{ik} must be found from

$$\sum_{k=1}^{r-1} B_{ik} = 1 \tag{10}$$

and

$$\sum_{i=1}^{r-1} \left[D^o_{li} \left(\frac{\rho_{i+} - \bar{\rho}_i}{\rho_{l+} - \bar{\rho}_l} \right) - \delta_{il}\sigma_k \right] B_{ik} = 0 \tag{11}$$

The above solution reduces to the usual binary case (*4, 6*). For the ternary case one obtains

$$\sigma_1 = \frac{1}{2} (D^o_{11} + D^o_{22} + \sqrt{(D^o_{11} - D^o_{22})^2 + 4D^o_{12}\, D^o_{21}}) \tag{12a}$$

$$\sigma_2 = \frac{1}{2} (D^o_{11} + D^o_{22} - \sqrt{(D^o_{11} - D^o_{22})^2 + 4D^o_{12}\, D^o_{21}}) \tag{12b}$$

$$B_{11} = \left[\sigma_1 - D^o_{22} + D^o_{12} \cdot \left(\frac{\rho_{2+} - \rho_{2-}}{\rho_{1+} - \rho_{1-}} \right) \right] / (\sigma_1 - \sigma_2) \tag{13}$$

$$B_{12} = \left[\sigma_2 - D^o_{22} + D^o_{12} \cdot \left(\frac{\rho_{2+} - \rho_{2-}}{\rho_{1+} - \rho_{1-}} \right) \right] / (\sigma_2 - \sigma_1) \tag{14}$$

$$B_{21} = \left[\sigma_1 - D^o_{11} + D^o_{21} \cdot \left(\frac{\rho_{1+} - \rho_{1-}}{\rho_{2+} - \rho_{2-}} \right) \right] / (\sigma_1 - \sigma_2) \tag{15}$$

$$B_{22} = \left[\sigma_2 - D^o_{11} + D^o_{21} \cdot \left(\frac{\rho_{1+} - \rho_{1-}}{\rho_{2+} - \rho_{2-}} \right) \right] / (\sigma_2 - \sigma_1) \tag{16}$$

Corresponding results may be obtained for four and five component systems. For a greater number of components, the eigenvalues must be found by numerical techniques.

By similar extension, these multicomponent flux equations may be solved for other geometries and boundary conditions. If the initial and boundary conditions for the multicomponent system have the same form as the conditions for the binary case, the solutions appear as a sum of the binary forms, each multiplied by a normalizing factor. The diffusion coefficient of this binary counterpart is replaced by the appropriate eigenvalue of the characteristic equation. The ease of obtaining solutions of this type is a principal advantage of these flux equations.

NOTATION

- b = one-half cell height
- B_{ij} = normalization constants
- D^o_{ij} = diffusion coefficients with respect to volume average velocity
- $f(z), g_i(t)$ = functions for separation of variables, Equation (7)
- t = time
- $\mathbf{v}^o$ = volume average velocity
- z = vertical cell coordinate
- α = eigenvalue of solution, Equation (8)
- Δ = operator for separation of simultaneous differential equations, Equation (9)
- ρ_i = mass density of component i
- σ_k = eigenvalue of characteristic equation

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